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Weighted composition operators from Nevanlinna type spaces to weighted Bloch type spaces

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Abstract

In this paper, we characterize metrically compact weighted composition operators from Nevanlinna type spaces to weighted Bloch type spaces.

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$, $\mathbb{T}$ the boundary of $\mathbb{D}$ and $d\sigma$ denote the normalized Lebesgue measure on $\mathbb{T}$. The Nevanlinna class N is defined as the set of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{0\le r<1}\int_{\mathbb{T}}\log(1+|f(r\zeta)|)\,d\sigma(\zeta)<\infty.$$

It is well-known fact that every holomorphic function f in the class N has a finite non-tangential limit, denoted by f^*, almost everywhere on $\mathbb{T}$.

For each $1<p<\infty$, the *Nevanlinna type space* N^p is a subspace of the Nevanlinna class N defined as

$$N^p=\left\{f\in H(\mathbb{D})\,:\,\|f\|_{N^p}=\sup_{0\le r<1}\left(\int_{\mathbb{T}}(\log(1+|f(r\zeta)|))^p\,d\sigma(\zeta)\right)^{1/p}<\infty\right\}.$$

N^p is an F-space with respect to the translation-invariant metric $d_{N^p}(f,g)=\|f-g\|_{N^p}$. Moreover, the subharmonicity of $(\log(1+|f|))^p$ implies that $f\in N^p$ has the following growth estimation:

$$\log(1+|f(z)|)\le\frac{4^{1/p}\|f\|_{N^p}}{(1-|z|^2)^{1/p}}\tag{1.1}$$

for $z\in\mathbb{D}$. Thus the convergence in N^p gives the uniform convergence on compact subsets of $\mathbb{D}$. For more informations on Nevanlinna type spaces, see [1, 2, 3, 5, 6, 14].

Let ν be a continuous function (*weight*) on $\mathbb{D}$ such that $\nu(z)=\nu(|z|)$ for every $z\in\mathbb{D}$ and ν is normal, that is, there exist positive numbers η and τ, $0<\eta<\tau$, and $\delta\in[0,1)$ such that

$$\frac{\nu(r)}{(1-r)^\eta}\text{ is decreasing on }[\delta,1)\text{ and }\lim_{r\to1}\frac{\nu(r)}{(1-r)^\eta}=0;$$

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$$\frac{\nu(r)}{(1-r)^\tau} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1-r)^\tau} = \infty.$$

For such a weight ν, the *weighted Bloch-type space* $\mathcal{B}_\nu$ on $\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty.$$

The *little weighted Bloch-type space* $\mathcal{B}_{\nu,0}$ consists of all $f \in \mathcal{B}_\nu$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0.$$

Both spaces $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)|,$$

and $\mathcal{B}_{\nu,0}$ is a closed subspace of $\mathcal{B}_\nu$. The compactness of a closed subset $L \subset \mathcal{B}_{\nu,0}$ can be characterized as follows.

Lemma 1.1. A closed set L in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{B}_\nu}$ and satisfies

$$\lim_{|z|\to 1} \sup_{f\in L} \nu(z)|f'(z)| = 0.$$

This result for the case $\nu(z) = (1-|z|^2)$ was proved by Madigan and Matheson [7]. By a modification of their proof, we can prove the above lemma. Lemma 1.1 is a very useful tool in the study of the compactness of linear operators, when the range space is $\mathcal{B}_{\nu,0}$(see [7, 8, 9, 10], *etc*).

Let $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$, the *weighted composition operator* $W_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $W_{\psi,\varphi}f = \psi \cdot f \circ \varphi$ for $f \in H(\mathbb{D})$. It is of interest to provide function-theoretic characterizations involving ψ and φ of boundedness and compactness of $W_{\psi,\varphi}$ acting between different function spaces. Given two linear topological vector spaces X and Y, a linear operator $T : X \to Y$ is *metrically bounded* if there exists a constant $C > 0$ such that $d_Y(Tf, 0) \le C d_X(f, 0)$ for $f \in X$. In general, the boundedness of T and the metrical boundedness of T do not coincide. However, if X and Y are Banach spaces, then the metrical boundedness of T coincides with the boundedness of T. For more informations on the metrical boundedness, we can refer to papers [2, 3]. Also recall that a linear operator $T : X \to Y$ is *bounded with respect to metric balls* if it takes every metric ball in X into a metric ball in Y. Since a metric ball is also a bounded set, so boundedness with respect to metric balls also coincides with metrical boundedness, if X and Y are Banach spaces. For the compactness of linear operators, for example, composition operators or weighted composition operators on these spaces, we use the metrical compactness. Namely, $T : X \to Y$ is *metrically compact* if it takes every metric ball in X into a relatively compact subset in Y. These operators on Nevanlinna type spaces have been studied by several authors (see [1, 2, 3, 8, 9, 10, 11, 12, 13]). Motivated by their work, in this paper, we will give characterizations for the boundedness with metric balls and the metrical compactness of $W_{\psi,\varphi}$ acting from Nevanlinna type spaces into weighted Bloch-type spaces.

2 Boundedness and compactness

We first give sufficient conditions for $W_{\psi,\varphi}$ to be bounded with respect to metric balls.

Proposition 2.1. If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ such that

$$\sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \infty \tag{2.1}$$

and

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \infty \tag{2.2}$$

for every $c > 0$, then $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls.

Proof. For $z \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, we have

$$1 - |z + (1-|z|)\zeta/2|^2 \geq 1 - (1+|z|)^2/4 \geq (1-|z|^2)/4.$$

Thus by Cauchy's integral formula and (1.1), we obtain that

$$(1-|z|^2)|f'(z)| \leq \frac{2}{\pi}\int_{\mathbb{T}} |f(z+(1-|z|)\zeta/2)|\,|d\zeta| \leq 4\exp\left\{\frac{4^{1+1/p}\|f\|_{N^p}}{(1-|z|^2)^{1/p}}\right\} \tag{2.3}$$

for each $z \in \mathbb{D}$. Hence we have that

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathcal{B}_\nu} &= |\psi(0)f(\varphi(0))| + \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z))| \\ &\leq |\psi(0)|\exp\left\{\frac{4^{1/p}\|f\|_{N^p}}{(1-|\varphi(0)|^2)^{1/p}}\right\} + \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)|\exp\left\{\frac{4^{1/p}||f||_{N^p}}{(1-|\varphi(z)|^2)^{1/p}}\right\} \\ &\quad + 4\sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{4^{1+1/p}||f||_{N^p}}{(1-|\varphi(z)|^2)^{1/p}}\right\}.\end{aligned}$$

Combining the above inequality and condition (2.1) and condition (2.2), we see that $W_{\psi,\varphi}$ takes every metric ball in N^p into a metric ball in $\mathcal{B}_\nu$, namely $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls. Q.E.D.

Proposition 2.2. If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ satisfying the conditions (2.1) and (2.2) for every $c > 0$, then $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded.

Proof. Suppose that conditions (2.1) and (2.2) hold, then proceeding as in the proof of Proposition 2.1, we see that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. Since N^p is an F-space, hence by the closed graph theorem, we have that $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is also bounded. Q.E.D.

The main result of this paper, characterizes the metrical compactness of $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$. In fact, we prove that boundedness of $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ with respect to metric balls is equivalent to metrically compactness of $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$.
To prove the main result, we need the following lemma.

Lemma 2.3. Suppose that $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. Then $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is metrically compact if and only if for any sequence $\{f_j\}$ in N^p with $\sup_j \|f_j\|_{N^p} \leq K$ and which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\{W_{\psi,\varphi} f_j\}$ converges to zero in $\mathcal{B}_\nu$.

Proof. This is an extension of a well-known result on the compactness of weighted composition operators on holomorphic function spaces. By (1.1), we see that any metrical bounded sequence in N^p form a normal family. Hence an argument by using the Montel theorem proves this lemma proceeding on the same lines as the proof of Proposition 3.11 in [4]. Q.E.D.

Theorem 2.4. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following conditions are equivalent;

(i) $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls,

(ii) $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is metrically compact,

(iii) $\psi \in \mathcal{B}_\nu$,

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi(z)\varphi'(z)| < \infty, \tag{2.4}$$

$$\lim_{|\varphi(z)|\to 1} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0 \tag{2.5}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0 \tag{2.6}$$

for any $c > 0$.

Proof. Since $\mathcal{B}_\nu$ is a Banach space, so each bounded set is also in a metric ball in $\mathcal{B}_\nu$. Hence the metrical compactness of $W_{\psi,\varphi}$ from N^p to $\mathcal{B}_\nu$ gives the boundedness with respect to metric balls. So implication (ii) $\Rightarrow$ (i) holds.

Now we will prove (iii) $\Rightarrow$ (ii). Assume that $\psi \in \mathcal{B}_\nu$ and that the conditions (2.4), (2.5) and (2.6) hold for every $c > 0$. Let $\varepsilon > 0$ be arbitrary. Then we can choose $r_0 \in (0,1)$ such that

$$\sup_{|\varphi(z)|>r_0} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \varepsilon$$

and

$$\sup_{|\varphi(z)|>r_0} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \varepsilon$$

for any $c > 0$. Choose a sequence $\{f_j\}$ in N^p such that $\sup_j \|f_j\|_{N^p} \leq K$ and $\{f_j\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Since $\psi \in \mathcal{B}_\nu$ and (2.5) imply (2.2), and (2.4) and (2.6) imply (2.1), so by Proposition 2.2, we see that $W_{\psi,\varphi}(N^p) \subset \mathcal{B}_\nu$. The assumptions $\psi \in \mathcal{B}_\nu$ and (2.4) also imply that

$$\sup_{|\varphi(z)|\leq r_0} \nu(z)|\psi'(z)||f_j(\varphi(z))| \leq \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)| \cdot \max_{|w|\leq r_0} |f_j(w)| \to 0$$

and

$$\sup_{|\varphi(z)|\le r_0} \nu(z)|\psi(z)\varphi'(z)||f_j'(\varphi(z))| \le \sup_{z\in\mathbb{D}} \nu(z)|\psi(z)\varphi'(z)| \cdot \max_{|w|\le r_0} |f_j'(w)| \to 0$$

as $j\to\infty$. Therefore, we have

$$\sup_{|\varphi(z)|\le r_0} \nu(z)|(W_{\psi,\varphi}f_j)'(z)| \to 0$$

as $j\to\infty$. On the other hand, it follows from (1.1) and (2.3) that

$$\begin{aligned}\sup_{|\varphi(z)|> r_0} \nu(z)|(W_{\psi,\varphi}f)'(z)| \le{}& \sup_{|\varphi(z)|> r_0} \nu(z)|\psi'(z)| \exp\left\{\frac{4^{1/p}K}{(1-|\varphi(z)|^2)^{1/p}}\right\} \\ &+ 4 \sup_{|\varphi(z)|> r_0} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{4^{1+1/p}K}{(1-|\varphi(z)|^2)^{1/p}}\right\} < \varepsilon.\end{aligned}$$

Thus we see that

$$\limsup_{j\to\infty} \|W_{\psi,\varphi}f_j\|_{\mathcal{B}_\nu} < \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we have that $\|W_{\psi,\varphi}f_j\|_{\mathcal{B}_\nu}\to 0$ as $j\to\infty$. Hence Lemma 2.3 shows that $W_{\psi,\varphi}: N^p \to \mathcal{B}_\nu$ is metrically compact.

Finally we prove the implication (i) $\Rightarrow$ (iii). By taking the constant function $f(z)=1$ in N^p, we have that $\psi\in\mathcal{B}_\nu$. Again by taking $f(z)=z$ in N^p and using facts that $|\varphi(z)|<1$ and $\psi\in\mathcal{B}_\nu$, we have that

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi(z)\varphi'(z)| < \infty.$$

Fix $c>0$ and put $w=\varphi(z)$. We define the following function

$$f_w(v) = \left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\} \exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{1/p}\right\}.$$

Then we can easily see that the family $\{f_w\}$ forms a metric ball in N^p. Also we have that

$$\begin{aligned}f_w'(v) ={}& \left\{\frac{6c\overline{w}}{p}\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{1/p} + 6\overline{w}\right\}\left\{\frac{1-|w|^2}{(1-\overline{w}v)^3} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\} \\ &\times \exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{1/p}\right\}.\end{aligned}$$

Since $\{W_{\psi,\varphi}f_w\}$ is a metric ball in $\mathcal{B}_\nu$, there is a positive constant C which is independent of $w=\varphi(z)$ such that $\|W_{\psi,\varphi}f_w\|_{\mathcal{B}_\nu}\le C$. Thus by using the fact that $f_w'(w)=0$, we obtain that

$$\begin{aligned}C &\ge \nu(z)|(W_{\psi,\varphi}f_w)'(z)| \\ &= \frac{\nu(z)|\psi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\},\end{aligned}$$

and so

$$\nu(z)|\psi'(z)|\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}\leq C(1-|\varphi(z)|^2).$$

Taking limit as $|\varphi(z)|\to 1$ on both sides of the above inequality, we get (2.5). Again fix $c>0$, put $w=\varphi(z)$ and consider the following function

$$h_w(v)=\left\{\frac{1-|w|^2}{(1-\overline{w}v)^2}-\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2}-2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{1/p}\right\}.$$

Then once again we can easily see that $\{h_w\}$ forms a metric ball in N^p and $h_w(w)=0$. Also we have that

$$\begin{aligned} h_w'(v) &= \left[\overline{w}\left\{2\frac{1-|w|^2}{(1-\overline{w}v)^3}-3\frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\}+\frac{6c}{p}\overline{w}\left\{\frac{1-|w|^2}{(1-\overline{w}v)^2}-\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right. \\ &\quad\times\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2}-2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{-1+1/p}\left.\left\{\frac{1-|w|^2}{(1-\overline{w}v)^3}-\frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\}\right] \\ &\quad\times\quad\exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2}-2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}^{1/p}\right\}. \end{aligned}$$

Therefore, we have that

$$h_w'(w)=\frac{-\overline{w}}{(1-|w|^2)^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}.$$

Since $\{W_{\psi,\varphi}h_w\}$ is a metric ball in $\mathcal{B}_\nu$, there is a positive constant C which is independent of $w=\varphi(z)$ such that $\|W_{\psi,\varphi}h_w\|_{\mathcal{B}_\nu}\leq C$. Thus we obtain that

$$\begin{aligned} C &\geq \nu(z)|(W_{\psi,\varphi}f_w)'(z)| \\ &= \frac{\nu(z)|\psi(z)\varphi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}, \end{aligned}$$

and so

$$\frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}\leq\frac{Cp(1-|\varphi(z)|^2)}{|\varphi(z)|}.$$

Taking limit as $|\varphi(z)|\to 1$ on both sides of the above inequality, we get (2.6). This completes the proof. Q.E.D.

Next we will investigate the compactness of operators $W_{\psi,\varphi}:N^p\to\mathcal{B}_{\nu,0}$.

Proposition 2.5. If $\psi\in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ such that

$$\lim_{|z|\to 1}\nu(z)|\psi'(z)|\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}=0 \tag{2.7}$$

$$\lim_{|z|\to 1}\frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}=0 \tag{2.8}$$

for all $c>0$, then $W_{\psi,\varphi}:N^p\to\mathcal{B}_{\nu,0}$ is bounded with respect to metric balls. Also we obtain that $W_{\psi,\varphi}(N^p)\subset\mathcal{B}_{\nu,0}$.

Proof. Since $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls by Proposition 2.1, we only need to prove that $W_{\psi,\varphi}(L) \subset \mathcal{B}_{\nu,0}$ for any metric balls L in N^p. However, the inequalities (1.1) and (2.3) show that

$$\nu(z)|(W_{\psi,\varphi}f)'(z)| \le \nu(z)|\varphi'(z)| \exp\left\{\frac{4^{1/p}\|f\|_{N^p}}{(1-|\varphi(z)|^2)^{1/p}}\right\} + 4\frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{4^{1+1/p}\|f\|_{N^+}}{(1-|\varphi(z)|^2)^{1/p}}\right\}$$

which holds for each $f \in L$. Thus the conditions (2.7) and (2.8) imply that $\nu(z)|(W_{\psi,\varphi}f)'(z)| \to 0$ as $|z| \to 1$. This completes the proof. Q.E.D.

Theorem 2.6. Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following conditions are equivalent;

(i) $W_{\psi,\varphi} : N^p \to \mathcal{B}_{\nu,0}$ is bounded with respect to metric balls,

(ii) $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls, $\psi \in \mathcal{B}_{\nu,0}$ and $\lim_{|z|\to 1} \nu(z)|\psi(z)\varphi'(z)| = 0$,

(iii) $W_{\psi,\varphi} : N^p \to \mathcal{B}_{\nu,0}$ is metrically compact,

(vii) ψ and φ satisfy the conditions (2.7) and (2.8).

Proof. The implication (iii) $\Rightarrow$ (i) is obvious. Also we can easily see that (i) $\Rightarrow$ (ii) are holds. In fact, we may consider the function $f(z) = 1$ in N^p. Then $\|f\|_{N^p} = \log 2$, and so f is in some metric balls in N^p. This shows that $\psi \in \mathcal{B}_{\nu,0}$. Again, consider the function $f(z) = z$ in N^p. Once again $\|f\|_{N^p} \le \log 2$, and so f is in some metric balls in N^p. Thus using $\psi \in \mathcal{B}_{\nu,0}$, we can shows that $\lim_{|z|\to 1} \nu(z)|\psi(z)\varphi'(z)| = 0$.

To prove (ii) $\Rightarrow$ (iv), we take a sequence $\{z_j\}$ in $\mathbb{D}$ with $|z_j| \to 1$ as $j \to \infty$. Then

$$\begin{aligned}&\limsup_{|z|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\}\\ &= \lim_{j\to\infty} \frac{\nu(z_j)|\psi(z_j)\varphi'(z_j)|}{1-|\varphi(z_j)|^2}\exp\left\{\frac{c}{(1-|\varphi(z_j)|^2)^{1/p}}\right\}.\end{aligned} \tag{2.9}$$

If $\sup_{j\ge 1}|\varphi(z_j)| < 1$, then the assumption $\psi \in \mathcal{B}_{\nu,0}$ implies that the right limit in the equation (2.9) is equal to 0, and so we obtain the condition (2.7). If $\sup_{j\ge 1}|\varphi(z_j)| = 1$, then we can choose a subsequence $\{z_{j_k}\} \subset \{z_j\}$ such that $|\varphi(z_{j_k})| \to 1$ as $k \to \infty$. Since $W_{\psi,\varphi} : N^p \to \mathcal{B}_\nu$ is bounded with respect to metric balls, by Theorem 2.4, ψ and φ satisfy

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0 \tag{2.10}$$

for any $c > 0$. By (2.9) and (2.10) we have that

$$\begin{aligned}
&\limsup_{|z|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} \\
&= \lim_{k\to\infty} \frac{\nu(z_{j_k})|\psi(z_j)\varphi'(z_{j_k})|}{1-|\varphi(z_{j_k})|^2} \exp\left\{\frac{c}{(1-|\varphi(z_{j_k})|^2)^{1/p}}\right\} \\
&\le \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{(1-|\varphi(z)|^2)^{1/p}}\right\} = 0.
\end{aligned}$$

This implies that (2.7) holds. Similarly, we can show that (2.8) also holds.

Finally we will prove the implication (iv) $\Rightarrow$ (iii). Take any metric ball L_{N^p} in N^p. Then there is a constant $K > 0$ such that $\|f\|_{N^p} \le K$ for any $f \in L_{N^p}$. For any $f \in L_{N^p}$ and $z \in \mathbb{D}$ we have that

$$\begin{aligned}
\nu(z)|(W_{\psi,\varphi}f)'(z)| &\le \nu(z)|\psi'(z)| \exp\left\{\frac{4^{1/p}K}{(1-|\varphi(z)|^2)^{1/p}}\right\} \\
&+ \quad 4\frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{4^{1+1/p}K}{1-|\varphi(z)|^2}\right\}.
\end{aligned}$$

Combining this with (2.7) and (2.8), we obtain

$$\lim_{|z|\to 1} \sup_{f\in L_{N^p}} \nu(z)|(W_{\psi,\varphi}f)'(z)| = 0,$$

and so Lemma 1.1 shows that $W_{\psi,\varphi}(L_{N^p})$ is compact in $\mathcal{B}_{\nu,0}$ for any metric ball L_{N^p}. This means that $W_{\psi,\varphi} : N^p \to \mathcal{B}_{\nu,0}$ is metrically compact. The proof is accomplished. Q.E.D.

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